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1995 J. Phys. A: Math. Gen. 28 L85

(http://iopscience.iop.org/0305-4470/28/3/005)

J. Phys. A: Math. Gen. 28 (1995) L85–L90. Printed in the UK

LETTER TO THE EDITOR

Dynamical r-matrices for some nonlinear oscillators

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Received 17 October 1994

Abstract. Dynamical r-matrices for the Rosochatius and Wojciechowski systems are found. They appear as a product of reduction from the constant r-matrices for the coupled Neumann and the Garnier system, respectively. They serve also as alternative r-matrices for the Neumann system and the anharmonic oscillator (respectively), for which the constant r-matrices are known.

The theory of dynamical r-matrices, as applied to finite-dimensional systems in classical mechanics, was dirst developed in 1990 [1] and has become increasingly popular since the discovery of r-matrices for the Calogero–Moser models [2]. It is still in the phase of collecting and classifying examples, see [3–5], and the goal of the present letter is to present two new ones, namely, the Wojciechowski system [6] and the Rosochatius system [7–11]. Our method of deriving the r-matrix structure for these models is very simple; nevertheless the results (equations (11), (21) below) are, to the author's knowledge, new. The method is based on the fact that these two systems may be viewed as reductions of two more general systems, the Garnier system [12–15] and the coupled Neumann system [16], respectively. The main message of the present letter reads: *reductions often lead to dynamical r-matrices*. It is, however, worth noting that the two simpler reductions of the above-mentioned systems, the anharmonic oscillator [12–15] and the Neumann system proper [8, 17–19] are also known to admit constant (i.e. independent of dynamical variables) r-matrix structures [14, 15, 19]. The relations between the old (constant) and the new (dynamical) r-matrices for these systems remains to be clarified.

We first turn our attention to the Wojciechowski system and the anharmonic oscillator. Here we consider the interplay between the r-matrix structures of the following systems: the Garnier system (G) and its two reductions, namely the anharmonic oscillator (AO) and the Wojciechowski system (W).

We begin with the Wojciechowski system. It is a Hamiltonian system in the phase space $\mathbb{R}^{2N}\{q, v\}$ equipped with a canonical Poisson bracket

$$\{v_k, q_j\} = \delta_{kj} \tag{1}$$

(here and below the brackets between the coordinate functions not written down explicitly are supposed to be equal to zero). The Hamiltonian function is

$$H^{W}(q, v) = \tfrac{1}{2}\langle v, v\rangle + \tfrac{1}{2}\langle \Omega q, q\rangle + \tfrac{1}{2}\langle q, q\rangle^2 + \tfrac{1}{2}\sum_{k=1}^{N} \frac{m_k^2}{q_k^2}. \tag{2}$$

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where $\langle\ ,\ \rangle$ stands for the standard scalar product in $\mathbb{R}^N$, $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$, and ω_k, m_k^2 are real numbers.

For the case of distinct ω_k Wojciechowski found N independent integrals of this system

$$F_k^{\mathrm{W}}(q, v) = v_k^2 + \omega_k q_k^2 + \frac{m_k^2}{q_k^2} + q_k^2\langle q, q\rangle + \sum_{j\neq k} \frac{1}{\omega_j - \omega_k}\left((v_k q_j - v_j q_k)^2 + \frac{m_k^2}{q_k^2}q_j^2 + \frac{m_j^2}{q_j^2}q_k^2\right)$$

and proved their involutivity [6]. He proposed two different proofs, one based on a direct and tiresome calculation, and the second following from the (spectral-parameter independent) Lax pair representation for the flows with Hamiltonian functions F_k^{W}. The Lax representation including a spectral parameter was found in [6] only for the flow with $H^{\mathrm{W}} = \frac{1}{2}\sum_{k=1}^N F_k^{\mathrm{W}}$ itself

$$\dot{L}^{\mathrm{W}} = [L^{\mathrm{W}}, M^{\mathrm{W}}]$$

where $L^{\mathrm{W}}, M^{\mathrm{W}}$ are $(N+1)\times(N+1)$ matrices depending on the phase space variables q, v and a spectral parameter λ

$$L^{\mathrm{W}}(q, v, \lambda) = \begin{pmatrix} \frac{1}{2}\lambda^2 E + \Omega + qq^{\mathrm{T}} & \lambda q + v + \mathrm{i}\dfrac{m}{q} \\ -\lambda q^{\mathrm{T}} + v^{\mathrm{T}} - \mathrm{i}\left(\dfrac{m}{q}\right)^{\mathrm{T}} & -\frac{1}{2}\lambda^2 - q^{\mathrm{T}}q \end{pmatrix}$$
$$M^{\mathrm{W}}(q, \lambda) = \begin{pmatrix} \frac{1}{2}\lambda E - \mathrm{i}Q & q \\ -q^{\mathrm{T}} & -\frac{1}{2}\lambda \end{pmatrix}. \tag{3}$$

Here E stands for the $N \times N$ identity matrix, and

$$\frac{m}{q} = \left(\frac{m_1}{q_1}, \ldots, \frac{m_N}{q_N}\right)^{\mathrm{T}} \qquad Q = \mathrm{diag}\left(\frac{m_1}{q_1^2}, \ldots, \frac{m_N}{q_N^2}\right).$$

The third proof of involutivity could be based on the r-matrix structure for the Lax matrix L^{W}, but such a structure was not found in [6]. It is one of the goals of the present letter to present explicitly this structure.

To this end consider the Wojciechowski system as a system in an extended phase space $\mathbb{C}^{4N}\{q, \varphi, v, m\}$ with a canonical Poisson bracket

$$\{v_k, q_j\} = \{m_k, \varphi_j\} = \delta_{kj} \tag{4}$$

with the Hamiltonian function $\widetilde{H}^{\mathrm{W}}(q, \varphi, v, m) = H^{\mathrm{W}}(q, v)$. Since φ_k are cyclic variables, the corresponding momenta m_k are the integrals of motion, so that the original Wojciechowski system is a restriction of the extended one to the common level set of m_k's, $1 \leqslant k \leqslant N$. Now perform the following change of variables:

$$x_k = q_k \mathrm{e}^{\mathrm{i}\varphi_k} \qquad \xi_k = q_k \mathrm{e}^{-\mathrm{i}\varphi_k} \qquad p_k = \left(v_k + \mathrm{i}\frac{m_k}{q_k}\right)\mathrm{e}^{\mathrm{i}\varphi_k} \qquad \pi_k = \left(v_k - \mathrm{i}\frac{m_k}{q_k}\right)\mathrm{e}^{-\mathrm{i}\varphi_k}. \tag{5}$$

It is easy to check that this change of variables is canonical (of valence 2), i.e. the Poisson brackets in new coordinates look like

$$\{p_k, \xi_j\} = \{\pi_k, x_j\} = 2\delta_{kj}. \tag{6}$$

The Hamiltonian function in the new variables takes the form

$$H(x, \xi, p, \pi) = \tfrac{1}{2}\langle p, \pi\rangle + \tfrac{1}{2}\langle \Omega x, \xi\rangle + \tfrac{1}{2}\langle x, \xi\rangle^2$$

which is, up to a factor $\frac{1}{2}$, a Hamiltonian function of the Garnier system. It is well known [13] that the Garnier system admits a Lax representation

$$\dot{L}^{\mathrm{G}} = [L^{\mathrm{G}}, M^{\mathrm{G}}]$$

with the matrices

$$\begin{aligned} L^{\mathrm{G}}(x,\xi,p,\pi,\lambda) &= \begin{pmatrix} \frac{1}{2}\lambda^2 E + \Omega + x\xi^{\mathrm{T}} & \lambda x + p \\ -\lambda\xi^{\mathrm{T}} + \pi^{\mathrm{T}} & -\frac{1}{2}\lambda^2 - \xi^{\mathrm{T}} x \end{pmatrix} \\ M^{\mathrm{G}}(x,\xi,\lambda) &= \begin{pmatrix} \frac{1}{2}\lambda E & x \\ -\xi^{\mathrm{T}} & -\frac{1}{2}\lambda \end{pmatrix}. \end{aligned} \tag{7}$$

The Lax matrix L^{G} under the Poisson bracket (6) satisfies the relation [14, 15]

$$\{L^{\mathrm{G}}(\lambda) \overset{\otimes}{,} L^{\mathrm{G}}(\mu)\} = \left[\frac{2\Pi}{\lambda-\mu}, I \bigotimes L^{\mathrm{G}}(\mu) + L^{\mathrm{G}}(\lambda) \bigotimes I\right] \tag{8}$$

where $\Pi = \sum_{j,k=1}^{N+1} E_{jk} \otimes E_{kj}$.

The connection between L^{W}, L^{G} follows straightforwardly from (3), (7) and (5), and reads

$$L^{\mathrm{G}} = \mathrm{e}^{\mathrm{i}\Phi} L^{\mathrm{W}} \mathrm{e}^{-\mathrm{i}\Phi} \qquad \Phi = \mathrm{diag}(\varphi_1, \ldots, \varphi_N, 0). \tag{9}$$

Substituting equation (9) into (8) and multiplying the resulting relation by $\mathrm{e}^{-\mathrm{i}\Phi} \otimes \mathrm{e}^{-\mathrm{i}\Phi}$ from the left and by $\mathrm{e}^{\mathrm{i}\Phi} \otimes \mathrm{e}^{\mathrm{i}\Phi}$ from the right, one arrives at

$$\{L^{\mathrm{W}}(\lambda) \overset{\otimes}{,} L^{\mathrm{W}}(\mu)\} = \left[r^{\mathrm{W}}(\lambda,\mu), I \bigotimes L^{\mathrm{W}}(\mu)\right] + \left[\tilde{r}^{\mathrm{W}}(\lambda,\mu), L^{\mathrm{W}}(\lambda) \bigotimes I\right] \tag{10}$$

where

$$r^{\mathrm{W}}(\lambda,\mu) = \frac{2\Pi}{\lambda-\mu} - \mathrm{i}\{L^{\mathrm{W}}(\lambda) \overset{\otimes}{,} \Phi\}$$

and for r^{W} and all other r-matrices we set

$$\tilde{r}(\lambda,\mu) = -r^*(\lambda,\mu) = -\Pi r(\mu,\lambda)\Pi.$$

The expression $\{L^{\mathrm{W}}(\lambda) \overset{\otimes}{,} \Phi\}$ does not depend on φ variables, as it should, and the result reads

$$r^{\mathrm{W}}(\lambda,\mu) = \frac{2\Pi}{\lambda-\mu} + \sum_{k=1}^{N} \frac{1}{q_k}(E_{k,N+1} - E_{N+1,k}) \bigotimes E_{kk}. \tag{11}$$

It is interesting enough to note that $r^{\mathrm{W}}(\lambda,\mu)$ does not depend on the values of m_k's either, and therefore also serves as an r-matrix for the anharmonic oscillator, which is nothing other than a Wojciechowski system with $m_1^2 = \ldots = m_N^2 = 0$. It is known, however [14, 15], that the Lax matrix for the anharmonic oscillator

$$L^{\mathrm{AO}}(q,v,\lambda) = \begin{pmatrix} \frac{1}{2}\lambda^2 E + \Omega + qq^{\mathrm{T}} & \lambda q + v \\ -\lambda q^{\mathrm{T}} + v^{\mathrm{T}} & -\frac{1}{2}\lambda^2 - q^{\mathrm{T}} q \end{pmatrix} \tag{12}$$

satisfies the r-matrix relation with a constant r-matrix

$$\{L^{\mathrm{AO}}(\lambda) \overset{\otimes}{,} L^{\mathrm{AO}}(\mu)\} = \left[r^{\mathrm{AO}}(\lambda,\mu), I \bigotimes L^{\mathrm{AO}}(\mu)\right] + \left[\tilde{r}^{\mathrm{AO}}(\lambda,\mu), L^{\mathrm{AO}}(\lambda) \bigotimes I\right] \tag{13}$$

$$r^{\mathrm{AO}}(\lambda,\mu) = \frac{\Pi}{\lambda-\mu} - \frac{1}{\lambda+\mu} \sum_{j,k=1}^{N+1} E_{jk} \bigotimes E_{jk}. \tag{14}$$

The relation between the r-matrices (12), (14) remains unclear.

We now turn our attention to the Rosochatius and the Neumann systems. We examine the interplay between the r-matrix structures of the coupled Neumann system (CN), and its two reductions: the Neumann system proper (N) and the Rosochatius system (R). All of them will be considered in the unconstrained version (the more convenient variant deals with their restrictions to the tangent bundle of the unit sphere in the configuration spaces).

We begin with the Rosochatius system, which is Hamiltonian in $\mathbb{R}^{2N}\{q,v\}$ with a Poisson bracket (1) and a Hamiltonian function

$$H^{\mathrm{R}}(q,v) = \tfrac{1}{2}(\langle v,v\rangle\langle q,q\rangle - \langle v,q\rangle^2) + \tfrac{1}{2}\langle\Omega q,q\rangle + \tfrac{1}{2}\langle q,q\rangle\sum_{k=1}^{N}\frac{m_k^2}{q_k^2}. \tag{15}$$

In the case of distinct ω_k's it posesses N independent integrals [8]

$$F_k^{\mathrm{R}}(q,v) = q_k^2 + \sum_{j\neq k}\frac{1}{\omega_k-\omega_j}\left((v_kq_j - v_jq_k)^2 + \frac{m_k^2}{q_k^2}q_j^2 + \frac{m_j^2}{q_j^2}q_k^2\right)$$

such that $\sum_{k=1}^{N} F_k^{\mathrm{R}} = \langle q,q\rangle$.

Moser proved their involutivity by constructing the (spectral-parameter independent) Lax pairs for all the flows with Hamiltonian functions F_k^{R} [8]. A spectral-parameter dependent Lax representation for the flow with $H^{\mathrm{R}} = \frac{1}{2}\sum_{k=1}^{N}\omega_kF_k^{\mathrm{R}}$ itself is easy to extract from [8]

$$\dot{L}^{\mathrm{R}} = [M^{\mathrm{R}}, L^{\mathrm{R}}]$$

where $L^{\mathrm{R}}, M^{\mathrm{R}}$ are $N\times N$ matrices depending on the phase space variables q, v and a spectral parameter λ

$$\begin{aligned} L^{\mathrm{R}}(q,v,\lambda) &= -\Omega + \lambda^{-1}\left(vq^{\mathrm{T}} - qv^{\mathrm{T}} + \mathrm{i}\left(\frac{m}{q}q^{\mathrm{T}} + q\left(\frac{m}{q}\right)^{\mathrm{T}}\right)\right) + \lambda^{-2}qq^{\mathrm{T}} \\ M^{\mathrm{R}}(q,\lambda) &= \lambda^{-1}qq^{\mathrm{T}} + \mathrm{i}\langle q,q\rangle Q \end{aligned} \tag{16}$$

(the meanings of the symbols Ω, $\frac{m}{q}$, and Q are the same as before).

In order to find the r-matrix structure for the Lax matrix (16) we proceed as before. Consider the Rosochatius system as a system in an extended phase space $\mathbb{C}^{4N}\{q,\varphi,v,m\}$ with the Poisson bracket (4), and the Hamiltonian function $\widetilde{H}^{\mathrm{R}}(q,\varphi,v,m) = H^{\mathrm{R}}(q,v) - \frac{1}{2}(\sum_{k=1}^{N} m_k)^2$. The variables φ_k are still cyclic, so that m_k are integrals of motion, and the original Rosochatius system is a restriction of the extended one to the common level set of m_k's, $1\leqslant k\leqslant N$. After the change of variables (5) the last Hamiltonian turns into

$$H(x,\xi,p,\pi) = \tfrac{1}{2}(\langle p,\pi\rangle\langle x,\xi\rangle - \langle p,\xi\rangle\langle x,\pi\rangle) + \tfrac{1}{2}\langle\Omega x,\xi\rangle$$

which is identical (up to a factor $\frac{1}{2}$) with a Hamiltonian function of the unconstrained version of the coupled Neumann system. It is known to admit a Lax representation [16]

$$\dot{L}^{\mathrm{CN}} = [M^{\mathrm{CN}}, L^{\mathrm{CN}}]$$

with

$$\begin{aligned} L^{\mathrm{CN}}(x,\xi,p,\pi,\lambda) &= -\Omega + \lambda^{-1}(p\xi^{\mathrm{T}} - x\pi^{\mathrm{T}}) + \lambda^{-2}x\xi^{\mathrm{T}} \\ M^{\mathrm{CN}}(x,\xi,\lambda) &= \lambda^{-1}x\xi^{\mathrm{T}}. \end{aligned} \tag{17}$$

It is easy to obtain the fundamental Poisson brackets for the Lax matrix (17) which follow from (6)

$$\{L^{\mathrm{CN}}(\lambda) \overset{\otimes}{,} L^{\mathrm{CN}}(\mu)\} = -\left[\frac{2\Pi}{\lambda-\mu}, I\bigotimes L^{\mathrm{CN}}(\mu) + L^{\mathrm{CN}}(\lambda)\bigotimes I\right] \tag{18}$$

where this time $\Pi = \sum_{j,k=1}^{N} E_{jk} \bigotimes E_{kj}$. The connection between L^{R}, L^{CN} follows directly from (16), (17) and (5), and reads

$$L^{\mathrm{CN}} = \mathrm{e}^{\mathrm{i}\Phi} L^{\mathrm{R}} \mathrm{e}^{-\mathrm{i}\Phi} \qquad \Phi = \mathrm{diag}(\varphi_1, \ldots, \varphi_N). \tag{19}$$

Substitute equation (19) into (18) and multiply the resulting equality by $\mathrm{e}^{-\mathrm{i}\Phi} \bigotimes \mathrm{e}^{-\mathrm{i}\Phi}$ from the left and by $\mathrm{e}^{\mathrm{i}\Phi} \bigotimes \mathrm{e}^{\mathrm{i}\Phi}$ from the right, to obtain

$$\{L^{\mathrm{R}}(\lambda) \overset{\otimes}{,} L^{\mathrm{R}}(\mu)\} = \left[r^{\mathrm{R}}(\lambda, \mu), I \bigotimes L^{\mathrm{R}}(\mu)\right] + \left[\tilde{r}^{\mathrm{R}}(\lambda, \mu), L^{\mathrm{R}}(\lambda) \bigotimes I\right] \tag{20}$$

where

$$r^{\mathrm{R}}(\lambda, \mu) = -\frac{2\Pi}{\lambda - \mu} - \mathrm{i}\{L^{\mathrm{R}}(\lambda) \overset{\otimes}{,} \Phi\}.$$

Again, this expression does not depend on φ variables

$$r^{\mathrm{R}}(\lambda, \mu) = -\frac{2\Pi}{\lambda - \mu} + \lambda^{-1} \sum_{j,k=1}^{N} \frac{q_j}{q_k} (E_{jk} + E_{kj}) \bigotimes E_{kk}. \tag{21}$$

This matrix does not depend on the values of m_k's either, so that (21) also serves as an r-matrix for the Neumann system, which is the $m_1^2 = \cdots = m_N^2 = 0$ particular case of the Rosochatius system. Again, the Lax matrix for the Neumann system

$$L^{\mathrm{N}}(q, v, \lambda) = -\Omega + \lambda^{-1}(vq^{\mathrm{T}} - qv^{\mathrm{T}}) + \lambda^{-2} qq^{\mathrm{T}} \tag{22}$$

is known [14, 19] to satisfy the relation

$$\{L^{\mathrm{N}}(\lambda) \overset{\otimes}{,} L^{\mathrm{N}}(\mu)\} = \left[r^{\mathrm{N}}(\lambda, \mu), I \bigotimes L^{\mathrm{N}}(\mu)\right] + \left[\tilde{r}^{\mathrm{N}}(\lambda, \mu), L^{\mathrm{N}}(\lambda) \bigotimes I\right] \tag{23}$$

with the constant r-matrix

$$r^{\mathrm{N}}(\lambda, \mu) = -\frac{\Pi}{\lambda - \mu} + \frac{1}{\lambda + \mu} \sum_{j,k=1}^{N} E_{jk} \bigotimes E_{jk}. \tag{24}$$

The relation between the r-matrices (21), (24) is not clear.

The author is happy to thank the Deutsche Forschungsgemeinschaft (DFG) for financial support.

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